On Generalized Mulholland Inequality and Dominance on Nilpotent Triangular Norms

Milan Petrík

Department of Mathematics,
Faculty of Engineering,
Czech University of Life Sciences,
Kamýcká 129,
165 21, Prague, Czech Republic
petrikm@tf.czu.cz

Institute of Computer Science,
Academy of Sciences of the Czech Republic,
Pod Vodárenskou věží 271/2,
182 07, Prague, Czech Republic
petrik@cs.cas.cz

October 29, 2017

Abstract

The paper is focused on the generalized Mulholland inequality and its connection with the dominance relation defined on the set of continuous Archimedean triangular norms. A counter-example is provided showing that the set of the solutions of the generalized Mulholland inequality is not closed with respect to compositions and that the dominance relation is not transitive on the set of nilpotent triangular norms.

Keywords: continuous Archimedean triangular norm, dominance relation, functional inequality, generalized Mulholland inequality, transitivity

MSC: Primary 26D07; Secondary 39B72, 26D15, 26A51, 03E72, 54E70.

1 Introduction

A triangular norm (or a t-norm) is a commutative, associative, and non-decreasing binary operation $\odot$ : $[0,1]^2 \to [0,1]$ with neutral element 1. This notion comes originally from the theory of probabilistic metric spaces [12, 26] where it describes the triangular inequality of probabilistic metrics. Examples of some prominent t-norms are listed in Table 1.
product: \( x \odot_P y = x \cdot y \)

Łukasiewicz t-norm: \( x \odot_L y = \max\{x + y - 1, 0\} \)

minimum: \( x \odot_M y = \min\{x, y\} \)

drastic t-norm: \( x \odot_D y = \begin{cases} 0 & \text{if } x, y < 1 \\ \min\{x, y\} & \text{otherwise} \end{cases} \)

Table 1: Prototypical examples of triangular norms.

The notion of a t-norm has, however, its importance also in other areas of mathematics. These operations play, for example, the role of the logical conjunction in the semantics of the basic logic [5, 6] and the monoidal t-norm based logic [3] which are both prototypical logics of graded truth (fuzzy logics). On the other hand, many t-norms are actually copulas [8, Definition 9.4] and thus they can be used to express dependence of random variables.

Dominance is a binary relation defined on the set of t-norms (in general it can be defined on any set of n-ary operations). A t-norm \( \odot_1 \) is said to dominate a t-norm \( \odot_2 \) (and we write \( \odot_1 \gg \odot_2 \)) if

\[
(x \odot_2 y) \odot_1 (u \odot_2 v) \geq (x \odot_1 u) \odot_2 (y \odot_1 v)
\]

holds for every \( x, y, u, v \in [0, 1] \). It was Tardiff who has started to study dominance of t-norms as he had recognized that this relation plays a crucial role when constructing Cartesian products of probabilistic metric spaces [30].

Since its introduction, there is an open question whether this relation is transitive, i.e., whether

\( \odot_1 \gg \odot_2 \text{ and } \odot_2 \gg \odot_3 \text{ implies } \odot_1 \gg \odot_3 \)

for every three t-norms \( \odot_1, \odot_2, \) and \( \odot_3 \). It can be proven that the dominance is both reflexive (this follows from the commutativity and the associativity of \( \odot_1 \) and \( \odot_2 \)) and anti-symmetric (put \( y = u = 1 \) in (1)) and that, for every t-norm \( \odot \), we have

\( \odot_M \gg \odot \gg \odot_D \).

Thus, if this relation is, moreover, transitive then it will become an order relation on the set of t-norms with a greatest and a least element. This question has been stated as an open problem [1, Problem 17] [26, Problem 12.11.3]:

**Problem 1.1** Is the dominance relation transitive, and hence a partial order, on the set of all t-norms? If not, for what subsets is this the case?

In the recent three decades many results have been done in this direction. In 1984, Sherwood has shown that the dominance relation is transitive on the class of Schweiser-Sklar t-norms [27].
In 2000, a remark has been stated in the monograph “Triangular norms” by Klement, Mesiar, and Pap [8, Example 6.17.v] from which it follows that the dominance relation is transitive on the class of Aczéli-Alsina t-norms, on the class of Dombi t-norms, and on the class of Yager t-norms.

In 2005, Sarkoci [22] has shown that also on the class of Frank t-norms and on the class of Hamacher t-norms the dominance relation is transitive.

Further, in the same year, Saminger-Platz, De Baets, and De Meyer [18] have shown that the dominance relation is transitive on the class of Mayor-Torrens t-norms and on the class of Dubois-Prade t-norms.

In 2008, Sarkoci has introduced a first counter-example showing that the dominance relation is, in general, not transitive on the set of continuous t-norms [21, 24]. Note that this counter-example does not involve continuous Archimedean t-norms. Therefore, the question, whether the dominance relation is transitive on the set of strict or nilpotent t-norms, had remained open.

In 2009, Saminger-Platz has shown that the dominance relation on the classes $T^8$, $T^9$, $T^{15}$, $T^{32}$, and $T^{34}$ (defined in the monograph “Associative Functions” [2] by Alsina, Frank, and Schweizer) is transitive [16].

In 2011, it has been proven with the help of a symbolic computation system [7] that on the class of Sugeno-Weber t-norms the dominance relation is transitive.

In 2014, Sarkoci has provided a characterization of the dominance relation on the set of ordinal sum t-norms with $\odot_L$ as the only summand operation and the set of ordinal sum t-norms with $\odot_P$ as the only summand operation [25].

Mulholland inequality is a functional inequality introduced by Mulholland in his paper from 1947 [13] as a generalization of Minkowski inequality. Recall that Minkowski inequality establishes the triangular inequality of $p$-norms (or $L^p$-norms). Mulholland, in his work, has replaced the power function in Minkowski inequality by an arbitrary increasing bijection $f: [0, \infty] \to [0, \infty]$. Thus, $f$ solves Mulholland inequality if

$$f^{-1}\left(\sum_{i=1}^{n} f(x_i + y_i)\right) \leq f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) + f^{-1}\left(\sum_{i=1}^{n} f(y_i)\right)$$

holds for every $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in [0, \infty]^n$.

Together with the introduction of the inequality, Mulholland has also provided the following sufficient condition [13, Theorem 1] (known as the Mulholland’s condition; for the proof see also Kuczma [9, Theorem VIII.8.1]):

**Theorem 1.2** Let $f: [0, \infty] \to [0, \infty]$ be an increasing bijection. If both $f$ and $\log \circ f \circ \exp$ are convex then $f$ solves Mulholland inequality.

The connection between Mulholland inequality and the dominance relation on strict t-norms has been revealed by Tardiff in 1984 [30]; it follows from his results that the dominance relation on the set of strict t-norms is transitive if,
and only if, the composition of two functions, that solve Mulholland inequality, solves the same inequality, again.

Until a recent time, the only known solutions of Mulholland inequality were delimited by the Mulholland’s condition [13] and it is easy to prove that a composition of two functions, that satisfy Mulholland’s condition, satisfies this condition, and thus also Mulholland inequality, again. However, since it was not known whether the Mulholland’s condition characterizes entirely all the solutions of Mulholland inequality, the mentioned result has not given an answer to the question whether the dominance relation is transitive on the set of strict t-norms, or not.

The result has come in 2015 when it has been shown that there are more solutions of Mulholland inequality, than the Mulholland’s condition delimits, and also that, in general, a composition of two such solutions does not solve the inequality, again [14]. Thanks to these results it has been shown that the dominance relation is not transitive on the set of strict t-norms [15].

**Generalized Mulholland inequality** [19] has been introduced in 2008 by Saminger-Platz, De Baets, and De Meyer in order to characterize, analogously to the previous case, the dominance relation on the set of all continuous Archimedean t-norms. Also in this case, the question, whether a composition of two solutions of the generalized inequality solves the same inequality, again, has remained open. In this paper, we are going to demonstrate that this generalized case can be treated in a way analogous to the previous case. As a result, the dominance relation is not transitive on the set of nilpotent triangular norms.

## 2 Triangular norms

This section provides a brief overview of the known results on continuous Archimedean t-norms. The proofs can be found, for example, in the book by Klement, Mesiar, and Pap [8]. Recall that the definition of a t-norm has been given already in the introduction.

A t-norm is said to be *continuous* if it is continuous as a two-variable real function. Observe that, while \(\oplus_P\), \(\oplus_L\), and \(\oplus_M\) are continuous, \(\oplus_D\) is not.

Let \(n \in \mathbb{N}\). We define the \(n\)-th natural power of \(x \in [0,1]\), according to a t-norm \(\odot\), as

\[
x^{(n)}_{\odot} = x \odot x \odot \ldots \odot x.
\]

A t-norm \(\odot\) is said to be *Archimedean* if, for every \(x, y \in [0,1]\), such that \(x < y\), there is \(n \in \mathbb{N}\) such that \(y^{(n)}_{\odot} < x\). A continuous t-norm \(\odot\) is Archimedean if, and only if, \(x \odot x < x\) for every \(x \in [0,1]\).

A continuous t-norm \(\odot\) is said to be *strict* if its restriction to \([0,1]^2\) is strictly increasing in both variables. Further, it is said to be *nilpotent* if, for every \(x \in [0,1]\) there is \(n \in \mathbb{N}\) such that \(x^{(n)}_{\odot} = 0\). It is known that a continuous Archimedean t-norm is either strict or nilpotent. A prototypical example of a
strict t-norm is $\circ_P$, a prototypical example of a nilpotent t-norm is $\circ_L$, an example of a continuous t-norm that is neither strict nor nilpotent is $\circ_M$.

A t-norm $\circ$ is strict if, and only if, there is a decreasing bijection $t: [0, 1] \rightarrow [0, \infty]$ such that, for every $x, y \in [0, 1],

$$x \circ y = t^{-1}(t(x) + t(y)).$$

The function $t$ is called the additive generator of the strict t-norm $\circ$. Remark that the additive generator of $\circ_P$ is the function $x \mapsto -\log x$.

A t-norm $\circ$ is nilpotent if, and only if, there is a decreasing injection $t: [0, 1] \rightarrow [0, \infty]$ with $t(0) < \infty$, such that, for every $x, y \in [0, 1],

$$x \circ y = t^{(-1)}(t(x) + t(y))$$

where $t^{(-1)}$ is the pseudo-inverse of $t$ defined by $t^{(-1)}(x) = t^{-1}(x)$ for every $x \in [0, t(0)]$ and by $t^{(-1)}(x) = 0$ for every $x \in [t(0), \infty]$.

The function $t$ is called the additive generator of the nilpotent t-norm $\circ$. Remark that the additive generator of $\circ_L$ is the function $x \mapsto 1 - x$.

Thus, every continuous Archimedean t-norm can be described by its additive generator; this generator is unique up to a multiplication by a positive real constant.

3 Generalized Mulholland inequality

In 2008, Saminger-Platz, De Baets, and De Meyer have introduced the following result [19, Theorem 1]:

**Theorem 3.1** Consider two continuous Archimedean t-norms $\circ_1$ and $\circ_2$ with additive generators $t_1$ and $t_2$, respectively. Then $\circ_1$ dominates $\circ_2$ if, and only
if, the functions $f = t_1 \circ t_2^{(-1)}$ and $f^{(-1)} = t_2 \circ t_1^{(-1)}$ satisfy, for every $x, y, u, v \in [0, t_2(0)]$,

$$f^{(-1)}(f(x + u) + f(y + v)) \leq f^{(-1)}(f(x) + f(y)) + f^{(-1)}(f(u) + f(v)). \quad \text{(GMI)}$$

We call (GMI) the generalized Mulholland inequality and if it is satisfied for every $x, y, u, v \in [0, t_2(0)]$ then we say that $f = t_1 \circ t_2^{(-1)}$ solves the generalized Mulholland inequality or, shortly, that is solves (GMI).

As we can see, the generalized Mulholland inequality gives a characterization of the dominance on continuous Archimedean t-norms in an analogous way as does Mulholland inequality in the case of strict t-norms.

Observe also that the functions $f$ and $f^{(-1)}$ from Theorem 3.1 satisfy the following assumptions, which we will utilize also in the sequel:

**Assumptions 3.2** Assume a function $f : [0, \infty] \to [0, \infty]$ and fixed values $d, e \in [0, \infty]$ such that:

1. $f(0) = 0$ and $f(d) = e$,
2. $f$ is continuous and strictly increasing on $[0, d]$,
3. $f(x) \geq e$ for $x \geq d$.

Assume, further, the function $f^{(-1)} : [0, \infty] \to [0, \infty]$ defined by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [0, e], \\ d & \text{otherwise.} \end{cases}$$

Figure 1 shows an example of functions $f$ and $f^{(-1)}$ that comply with Assumptions 3.2.
4 Geometric interpretation of the generalized Mulholland inequality

Let us consider functions $f$ and $f^{(-1)}$ according to Assumptions 3.2 and let us define the binary operation $\ast : [0, \infty]^2 \to [0, \infty]$, for every $x, y \in [0, \infty]$, by:

$$x \ast y = f^{(-1)}(f(x) + f(y)).$$

We call $\ast$ the bounded pseudo-addition generated by $f$.

With a help of this new operation we can rewrite (GMI) in a nicer form:

$$(x + u) \ast (y + v) \leq (x \ast y) + (u \ast v).$$

(2)

Notice that this new form, actually, states that a function $f$ solves (GMI) if, and only if, the classical addition dominates the bounded pseudo-addition generated by $f$.

The level set of a bounded pseudo-addition $\ast$ at the level $z \in [0, \infty]$ is the set

$$L_z = \{(x, y) \in [0, \infty]^2 \mid x \ast y = z\}.$$

The level cut of $\ast$ at the level $z \in [0, \infty]$ is the set

$$\Lambda_z = \{(x, y) \in [0, \infty]^2 \mid x \ast y \leq z\}.$$

The support of $\ast$ is the set

$$\text{supp} \ast = \{(x, y) \in [0, \infty]^2 \mid x \ast y < d\}.$$

The collection of all the level sets of $\ast$ is called the level set plot of $\ast$; an example is shown in Figure 2.

Note that the level sets and the level cuts of $\ast$ have the following structure.

• For $z < d$, $\Lambda_z$ is a convex set and $L_z$ is its border in $[0, \infty]^2$.
• For $z = d$, $\Lambda_z = [0, \infty]^2$ and $L_z$ is the complement of $\text{supp} \ast$.
• Finally, for $z > d$, $\Lambda_z = [0, \infty]^2$ and $L_z$ is an empty set.

Minkowski sum of two sets, $A, B \subseteq [0, \infty]^2$, is defined by

$$A + B = \{(x + u, y + v) \mid (x, y) \in A, (u, v) \in B\}.$$

Consider (2) and let $x \ast y = a$ and $u \ast v = b$ such that $a + b \leq d$. (Observe that for $a + b > d$ the generalized Mulholland inequality is always satisfied.) Then $(x, y) \in L_a$ and $(u, v) \in L_b$ as illustrated in Figure 3. The left-hand expression of (2) is the value of $\ast$ at the point $(x + u, y + v) = (x, y) + (u, v)$ and this value is supposed to be lower or equal to $a + b$; or, in other words, the point $(x + u, y + v)$ must be contained within $\Lambda_{a+b}$. We can imagine this as shifting of $\Lambda_a$ such that its bottom-left corner coincides with the level set $L_b$; the shifted
Figure 3: Geometric interpretation of the generalized Mulholland inequality.

$\Lambda_a$ must remain contained in the level cut $\Lambda_{a+b}$ as shown in Figure 3. Since all the possible additions of points $(x, y)$ and $(u, v)$ such that $x \ast y \leq a$ and $u \ast v \leq b$ can be obtained by $\Lambda_a + \Lambda_b$, we can state that (2) holds for every $x, y, u, v \in [0, \infty]$ if, and only if,

$$\Lambda_a + \Lambda_b \subseteq \Lambda_{a+b}.$$  

(3) holds for every $a, b \in [0, \infty]$. This way, we have obtained even more compact form of the generalized Mulholland inequality which is quantified by two variables only.

5 Sufficient condition by Saminger-Platz, De Baets, and De Meyer

This section presents a sufficient condition on functions that solves (GMI) which has been introduced by Saminger-Platz, De Baets, and De Meyer in 2008 [19, Theorem 1].

Following the terminology of Matkowski [10], we say that a function $f : [0, \infty] \to [0, \infty]$ is geometrically convex on an interval $]m, n[ \subseteq [0, \infty]$ if, for every $x, y \in ]m, n[$ and $\alpha \in [0, 1]$, we have

$$f \left( x^{1-\alpha} \cdot y^\alpha \right) \leq f^{1-\alpha}(x) \cdot f^\alpha(y).$$

Note that the function $f$ is geometrically convex on an interval $]m, n[ \subseteq [0, \infty]$ if, and only if, the function $\log \circ f \circ \exp$ is convex on the interval $]\log m, \log n[ \subseteq \mathbb{R}$.

In an analogy with the result by Sarkoci [23, Proposition 13], we introduce a characterization of geometric convexity on a given interval.
Proposition 5.1 Consider a function $f : [0, \infty] \rightarrow [0, \infty]$ and an interval $]m, n] \subseteq [0, \infty]$ such that $f$, restricted to $]m, n]$, is a continuous increasing injection. Then $f$ is geometrically convex on $]m, n]$ if, and only if, there exist sequences $(p_i)_{i \in \mathbb{N}}$ and $(q_i)_{i \in \mathbb{N}}$, with $p_i \geq 1$ and $q_i > 0$ for every $i \in \mathbb{N}$, such that $f(x) = \sup_{i \in \mathbb{N}} q_i x^{p_i}$ for every $x \in ]m, n]$.

The promised sufficient condition now follows. Notice that this condition is in an analogy with the Mulholland’s one presented in Theorem 1.2.

Theorem 5.2 Consider functions $f$ and $f^{(-1)}$ according to Assumptions 3.2 such that $f$ is

- convex on $]0, d[$,
- geometrically convex on $]0, d[$.

Then (GMI) holds for every $x, y, u, v \in [0, \infty]$, i.e., $f$ solves the generalized Mulholland inequality.

6 Counter-example

The sufficient condition presented in Theorem 5.2 is, however, not a necessary one, i.e., it does not delimit all the functions that solve (GMI). To demonstrate this fact, we introduce a class of functions which do not comply with the assumptions of Theorem 5.2 but which do solve (GMI).
Consider the following parametric class of functions. A function \( g: [0, \infty) \to [0, \infty) \) is given, for a parameter \( r \in \left[ 0, \frac{1}{2} \right] \) and for every \( x \in [0, \infty) \) by

\[
g(x) = \begin{cases} 
2rx & \text{if } x \in [0, \frac{1}{2}], \\
(2 - 2r)x - 1 + 2r & \text{if } x \in \left( \frac{1}{2}, 1 \right], \\
1 & \text{if } x > 1.
\end{cases}
\] (4)

This function, illustrated in Figure 4-left, is not geometrically convex. Indeed, it does not comply with the assumptions of Proposition 5.1. However, it does solve (GMI) as we are going to demonstrate. The pseudo-addition generated by \( g \) is given, for every \( x, y \in [0, \infty) \), by

\[
x \ast y = \begin{cases} 
x + y & \text{if } x + 1 \leq 1, \\
\frac{x + y + 1 - 2r}{2 - 2r} & \text{if } x \leq \frac{1}{2}, y \leq \frac{1}{2}, x + 1 > 1, \\
x + \frac{r}{1-r}y & \text{if } x > \frac{1}{2}, y \leq \frac{1}{2}, x + \frac{r}{1-r}y \leq 1, \\
\frac{x}{1-r} + y & \text{if } x \leq \frac{1}{2}, y > \frac{1}{2}, \frac{r}{1-r}x + y \leq 1, \\
x + y - \frac{1 - 2r}{2 - 2r} & \text{if } x > \frac{1}{2}, y > \frac{1}{2}, x + y \leq \frac{3 - 4r}{2 - 2r}, \\
1 & \text{otherwise.}
\end{cases}
\]

See an illustration of its level set plot in Figure 5-left.

Now take \( a, b \in [0, \infty) \) such that \( a \leq b \).

- If \( a + b > 1 \) then the generalized Mulholland inequality is trivially satisfied (confer with Figure 3-right.)

- If \( a + b \leq 1 \) then, necessarily, \( a \leq \frac{1}{2} \). In such a case \( \Lambda_a \) has the shape of a right-angled isosceles triangle—it is the minimal convex level cut for the given level \( a \). It is an easy but technical proof to show that, if \( \Lambda_a \) is of such a shape, and \( \Lambda_b \) and \( \Lambda_{a+b} \) are convex sets, then the condition described in Figure 3-left is satisfied.

### 7 Compositions of solutions of the generalized Mulholland inequality

This section is going to show that the set of solutions of the generalized Mulholland inequality is not closed with respect to compositions.

To vindicate this fact, let us take the function \( g \) defined by (4) and the function \( h: [0, \infty) \to [0, \infty) \) (see Figure 4-right) given, for every \( x \in [0, \infty) \), by:

\[
h(x) = \begin{cases} 
x^2 & \text{if } x \in [0, 1], \\
1 & \text{if } x > 1.
\end{cases}
\] (5)

It can be easily observed (confer with Proposition 5.1) that \( h \) satisfies the assumptions of Theorem 5.2 and thus does solve (GMI).
Figure 5: Left: Level set plot of the bounded pseudo-addition generated by the function $g$. Right: Level set plot of the bounded pseudo-addition generated by the function $g \circ h$.

However, while both $g$ and $h$ both solve (GMI), $g \circ h$ does not. Consider the bounded pseudo-addition $*_{g \circ h}$ generated by $g \circ h$ and compare its level set plot (illustrated in Figure 5-right) with the level set plot of the bounded pseudo-addition $*_g$ generated by $g$ (illustrated in Figure 5-left).

Observe that the level sets of $*_{g \circ h}$ are images of the level sets of $*_g$ under the mapping

$$T : [0, \infty)^2 \to [0, \infty)^2 : (x, y) \mapsto (\sqrt{x}, \sqrt{y}).$$

In supp $*_g$, the level sets are composed of line segments which either slopes at the angle $\frac{3}{4}\pi$ from the horizontal axis, or another constant angle, say $\phi$ (see Figure 5-left). The segments that slopes at the angle $\frac{3}{4}\pi$ are mapped by $T$ to parts of circles; the segments that slopes at the angle $\phi$ are mapped to parts of ellipses. The whole image is the level set plot of $*_{g \circ h}$ illustrated by Figure 5-right.

This level set plot, however, does not satisfy the condition described in Figure 3-left. Let us demonstrate this by drawing two auxiliary line segments to the level set plot of $*_g$ (depicted by gray dashed lines in Figure 6-left). Images of these auxiliary line segments are two parts of circles (depicted by gray dashed curves in in Figure 6-right). Let $b' \in [0, \infty]$ be the level of the level set of $*_g$ that contains the point $(\sqrt{1/2}, \sqrt{1/2})$. Observe that the level sets of $*_g$ in the levels $b'$ and 1 (emphasized in Figure 6-left) are mapped to the level sets of $*_{g \circ h}$ in the levels $b$ and 1 (emphasized in Figure 6-right).

Now, consider the level $a = 1 - b$ in $*_{g \circ h}$, and observe the corresponding level cut $\Lambda_a$ emphasized in Figure 6-right. As the point $(\sqrt{1/2}, \sqrt{1/2})$ is contained in $L_b$, the level cut $\Lambda_a$ shifted to this point should be fully contained in $\Lambda_1$. This is, however, not the case since the distance $c - 1$ is strictly smaller than the radius $a$. The reason is that $1 - b' = c' - 1$ but, since $b = \sqrt{b'}$, $c = \sqrt{c'}$, and since the square root is an increasing concave function, as demonstrated in Figure 7, we have $a = 1 - b > c - 1$. 

11
Transitivity of dominance on nilpotent t-norms

Analogously to the case of strict t-norms and Mulholland inequality, we can state the following result for continuous Archimedean t-norms and the generalized Mulholland inequality:

**Proposition 8.1** The relation of dominance is transitive on the set of continuous Archimedean t-norms if, and only if, the set of solutions of the generalized Mulholland inequality is closed with respect to compositions.

We can demonstrate the proof by taking three continuous Archimedean t-norms $\otimes_1$, $\otimes_2$, and $\otimes_3$ given, respectively, by their additive generators $t_1$, $t_2$, and $t_3$. Suppose that $\otimes_1 \gg \otimes_2$ and $\otimes_2 \gg \otimes_3$; this holds true if, and only if, $f_{1,2} = t_1 \circ t_2^{(-1)}$ and $f_{2,3} = t_2 \circ t_3^{(-1)}$ solve the generalized Mulholland inequality. Observe that $f_{1,3} = f_{1,2} \circ f_{2,3}$ is equal to $t_1 \circ t_3^{(-1)}$ on $[0, t_3((0)]$. This means that $\otimes_1 \gg \otimes_3$ if, and only if, $f_{1,3}$ solves the generalized Mulholland inequality.

An example of a triad of strict t-norms which disprove the transitivity of the dominance relation has been already demonstrated in a previous result [15, Theorem 10.3]. To obtain a triad of nilpotent t-norms, let us consider the following three additive generators given for every $x \in [0,1]$ by:

\[
\begin{align*}
t_1(x) &= \begin{cases} 
-2(1-r)x + 1 & \text{if } x \in [0, \frac{1}{2}], \\
-2r x + 2r, & \text{if } x \in [\frac{1}{2},1],
\end{cases} \\
t_2(x) &= 1 - x, \\
t_3(x) &= \sqrt{1 - x}
\end{align*}
\]

where $r \in ]0, \frac{1}{2}[$ is a parameter. These three functions generate, respectively, the three nilpotent t-norms $\otimes_1$, $\otimes_2$, and $\otimes_3$. Observe that $t_1 \circ t_2^{(-1)}$ is, actually,
Figure 7: Graph of the square root function.

the function $g$ given by (4). Since $g$ solves (GMI), we have $\circ_1 \gg \circ_2$. Observe further that $t_2 \circ t_3^{(-1)}$ is the function $h$ given by (5). Since also $h$ solves (GMI), we have $\circ_2 \gg \circ_3$. However, $t_1 \circ t_3^{(-1)} = g \circ h$ and, therefore, $\circ_1 \not\gg \circ_3$.

9 Conclusion

The paper has demonstrated that the set of solutions of the generalized Mulholland inequality is larger than the set delimited by the condition introduced by Saminger-Platz, De Baets, and De Meyer. Further, it has shown that the set of solutions of the generalized Mulholland inequality is not closed with respect to compositions and, consequently, that the dominance relation is not transitive on the set of nilpotent t-norms.

However, the question of characterization of all the solutions of the generalized Mulholland inequality still remains open.

Acknowledgments

This work was supported by the Czech Science Foundation under Project GJ15-07724Y.

The author would like to thank the anonymous referees for careful reading and correcting this paper.

References


